ELSEVIER

Journal of Geometry and Physics 45 (2003) 130–150

JOURNAL OF GEOMETRY AND PHYSICS

www.elsevier.com/locate/jgp

On the integrability of Bertrand curves and Razzaboni surfaces

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Received 2 April 2002

Abstract

Based on classical but apparently little known results due to Razzaboni, the integrable nature of Bertrand curves and their geodesic embedding in surfaces is discussed in the context of modern soliton theory. The existence of parallel Razzaboni surfaces which constitute the surface analogues of the classical offset Bertrand mates is recorded. It is shown that the natural geodesic coordinate systems on Razzaboni surfaces and their mates are related by a reciprocal transformation. The geodesic coordinate system on the Razzaboni transform generated by a Bäcklund transformation is given explicitly in terms of Razzaboni's pseudopotential obeying a compatible Frobenius system. The Razzaboni transformation and the duality transformation which links a Razzaboni surface and its mate are proven to commute. A canonical quantity introduced by Razzaboni is recognized as an invariant of the Razzaboni and duality transformations. Finally, Razzaboni surfaces are shown to be amenable to the Sym–Tafel formula.

MSC: 53A99; 35Q51

JGP SC: Differential geometry

Keywords: Integrability; Bertrand curves; Razzaboni surfaces; Solitons

1. Introduction

It is by now well established that classical differential geometry constitutes a repository of integrable classes of surfaces, i.e. classes of surfaces which are governed by integrable nonlinear systems. Amongst those are surfaces of constant Gaußian or mean curvature, isothermic and minimal surfaces, affine spheres and projective minimal surfaces. Thus, distinguished geometers such as Bianchi, Calapso, Darboux, Demoulin, Guichard, Jonas, Ribaucour and Weingarten investigated these classes in detail and, in particular, recorded

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PII: S0393-0440(02)00130-4

associated Bäcklund transformations and linear representations (see [1], and references therein). The natural coordinates on these surfaces are asymptotic, conjugate, conformal or curvature coordinates.

Integrable surfaces on which geodesics and their orthogonal trajectories form canonical coordinate systems seem to have attracted less attention by the geometers of the nineteenth century. Such surfaces may be regarded as being swept out by integrable binormal motions of inextensible curves. The most important surfaces of this kind are (Hasimoto) surfaces which are generated by motions of a curve the local speed of which is proportional to the local curvature of the curve. In 1906, Da Rios [2] was led to these particular motions in connection with an investigation of thin isolated vortex filaments traveling without stretching in an incompressible fluid. Remarkably, some 70 years later, Hasimoto [3] demonstrated that the pair of coupled nonlinear equations set down by Da Rios may be combined to the integrable nonlinear Schrödinger equation.

On use of a formulation with its origin in a kinematic study of hydrodynamics by Marris and Passman [4], particular toroidal Hasimoto surfaces have recently been shown to form nested constant pressure surfaces in steady hydrodynamics or, equivalently, magnetic surfaces in magnetohydrostatics [5,6]. In [7], the same formalism has led to the discovery of integrable surfaces which are spanned by a one-parameter family of geodesics of constant curvature or torsion. Thus, in terms of the equivalent notion of binormal motions of curves, the case of constant curvature has been related to an integrable extension of the Dym equation which may be linked to the modified modified Korteweg-de Vries (m^2KdV) equation via a reciprocal transformation. A variant of the integrable reduced Maxwell–Bloch system [8] which may be regarded as a generalization of the classical sine-Gordon and self-induced transparency (SIT) equations [9] has been shown to govern the binormal motion of curves of constant torsion.

Curves of constant curvature or torsion constitute particular Bertrand curves. Bertrand curves are well-studied classical curves and may be defined by their property that any Bertrand curve shares its principal normals with another Bertrand curve, sometimes referred to as Bertrand mate [10]. Accordingly, Bertrand mates represent particular examples of offset curves [11] which are used in computer-aided design (CAD) and computer-aided manufacture (CAM). The distance between a Bertrand curve and its mate measured along the principal normal is known to be constant. This particular offset property may be used to show that any surface which is spanned by a one-parameter family of geodesic Bertrand curves of the same 'kind' admits a parallel surface of the same type. Moreover, application of the Wahlquist–Estabrook prolongation technique [12,13] to the underlying nonlinear Gauß–Mainardi–Codazzi equations reveals that these surfaces are integrable.

It turns out that the above-mentioned class of surfaces which admits a geodesic embedding of Bertrand curves was studied in detail by Amilcare Razzaboni who was an assistant to Dini and a member of the Academy of Science of Bologna. In fact, in 1903, Razzaboni [14] derived a Bäcklund transformation for this class of surfaces and set down a coupled Riccati system which, in modern terminology, is nothing but a Lax pair for the underlying nonlinear Gauß–Mainardi–Codazzi equations. As Razzaboni noted, the latter may be cast into the form of a single nonlinear equation of fourth order. In connection with geodesic curves of constant curvature or torsion, Razzaboni even refers to earlier work [15] published in 1898 on what we may now call Razzaboni surfaces and also mentions a paper by Fibbi [16] which deals with the constant torsion case. Thus, it is evident that the reduced Maxwell–Bloch system and the (extended) Dym equation are implicitly contained in work of the nineteenth century.

The purpose of the present paper is twofold. On the one hand, it appears that Razzaboni's publications are little known and it is therefore desirable to make them accessible to a wider community. On the other hand, against the background of modern soliton theory, novel interesting results have emerged. Thus, we begin with a review of the notion of Bertrand curves and their offset curves. We then give the definition of Razzaboni surfaces and show that there exist dual parallel Razzaboni surfaces which constitute the surface analogues of the classical Bertrand mates. This 'duality' transformation seems to have escaped Razzaboni's attention. Moreover, we demonstrate that the natural geodesic coordinate systems parametrizing the geodesic Bertrand curves and their orthogonal trajectories on Razzaboni surfaces and their mates are related by a reciprocal transformation which induces an invariance of the Gauß–Mainardi–Codazzi equations. Reciprocal transformations have a long history. Accounts of reciprocal transformations and their application in continuum mechanics may be found in [17,18]. They also play an important role in the classification of so-called systems of hydrodynamic type [19–22].

In Section 4, we recall Razzaboni's Bäcklund transformation for Bertrand curves [23] which generalizes a result due to Demartres [24]. The Razzaboni transformation constitutes an extension to Bertrand curves of standard Bäcklund transformations for curves of constant curvature or torsion. The Bäcklund transformation for Razzaboni surfaces [14] is then presented. Razzaboni's transformation is formulated in terms of a Frobenius system for a pseudopotential ϕ and an arbitrary parameter k. This system is equivalent to a linear matrix system which constitutes a linear representation in the sense of soliton theory. In Section 5, we establish that this linear system, in which the Bäcklund parameter k plays the role of a 'spectral parameter', encapsulates the complete class of Razzaboni surfaces via the Sym–Tafel formula [25]. The latter is an important tool in the geometric study of integrable systems.

In the remainder of Section 4, we verify Razzaboni's Bäcklund transformation by constructing the natural geodesic coordinate system on the transform Σ' of a Razzaboni surface Σ. It turns out that the exact one-form which defines arc length of the Bertrand curves on Σ' may be integrated explicitly in terms of ϕ. Thus, in order to determine the associated geodesic coordinate system on Σ' no further integration is required. Moreover, an analogous statement may be made in the case of the Razzaboni mate Σ^*. This observation then leads to the result that the Razzaboni and duality transformations commute. It is also recorded that the duality transformation may be recovered from the Razzaboni transformation in a formal limit. A similar limit gives rise to a novel Bäcklund transformation for Razzaboni surfaces for which the binormals to the Bertrand curves and their transforms are pointwise orthogonal. Finally, Razzaboni's quantity which obeys the above-mentioned fourth-order equation and in terms of which the fundamental forms of Razzaboni surfaces may be expressed is shown to be an invariant of the Razzaboni and duality transformations.

2. Bertrand curves

In the present paper, we are concerned with curves and surfaces in Euclidean space $\mathbb{R}^3$. If a curve $\Gamma : \boldsymbol{r} = \boldsymbol{r}(s)$ is parametrized in terms of arc length then the orthonormal triad

$(\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b})$ consisting of the unit tangent vector $\boldsymbol{t} = \boldsymbol{r}_s$, the principal normal $\boldsymbol{n}$ and the binormal $\boldsymbol{b}$ varies along Γ according to the Serret–Frenet equations [26]:

$$\begin{pmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{pmatrix}_s = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{pmatrix}, \tag{2.1}$$

where the quantities κ and τ denote the curvature and torsion of the curve, respectively. An offset curve Γ^* along the principal normal is defined by

$$\boldsymbol{r}^* = \boldsymbol{r} + \alpha \boldsymbol{n}, \tag{2.2}$$

where α constitutes a prescribed function of s. If one demands that the parent curve Γ and its offset curve Γ^* occur on an equal footing then the principal normals to Γ and Γ^* must coincide, i.e.:

$$\boldsymbol{n}^* = \boldsymbol{n}. \tag{2.3}$$

This imposes constraints on the parent curve and the 'distance' function α. Thus, differentiation of (2.2) yields

$$\boldsymbol{r}^*_s = (1 - \alpha\kappa)\boldsymbol{t} + \alpha_s \boldsymbol{n} + \alpha\tau \boldsymbol{b}, \tag{2.4}$$

which, by virtue of $\boldsymbol{r}^*_s \cdot \boldsymbol{n}^* = 0$, has the important implication that the offset curve is at a constant distance α from the parent curve, i.e. $\alpha_s = 0$. The unit tangent vector $\boldsymbol{t}^*$ is therefore given by

$$\boldsymbol{t}^* = \frac{(1 - \alpha\kappa)\boldsymbol{t} + \alpha\tau \boldsymbol{b}}{D}, \quad D = \sqrt{(1 - \alpha\kappa)^2 + \alpha^2\tau^2}. \tag{2.5}$$

Further differentiation produces

$$\boldsymbol{t}^*_s = \left(\frac{1 - \alpha\kappa}{D}\right)_s \boldsymbol{t} + \frac{(1 - \alpha\kappa)\kappa - \alpha\tau^2}{D} \boldsymbol{n} + \left(\frac{\alpha\tau}{D}\right)_s \boldsymbol{b}. \tag{2.6}$$

The $\boldsymbol{t}$- and $\boldsymbol{b}$-components of the above are required to vanish since $\boldsymbol{t}^*_s \| \boldsymbol{n}^*$. It is readily verified that this requirement leads to the curvature–torsion relation:

$$\alpha\kappa + \beta\tau = 1, \tag{2.7}$$

where β constitutes a constant of integration. Curves for which there exist constants α and β such that (2.7) holds are known as Bertrand curves [10,11]. Accordingly, the following classical theorem holds.

Theorem 1 (The offset property of Bertrand curves). *A curve Γ admits an offset curve Γ^* which has the same principal normal as the parent curve if and only if Γ is a Bertrand curve, i.e.*:

$$\alpha\kappa + \beta\tau = 1 \tag{2.8}$$

for some constants α *and* β. *The offset curve* Γ^* *and its orthonormal triad* $(\boldsymbol{t}^*, \boldsymbol{n}^*, \boldsymbol{b}^*)$ *are related to* Γ *by*

$$\boldsymbol{r}^* = \boldsymbol{r} + \alpha\boldsymbol{n}, \qquad \boldsymbol{t}^* = \frac{\beta\boldsymbol{t} + \alpha\boldsymbol{b}}{\sqrt{\alpha^2+\beta^2}}, \qquad \boldsymbol{n}^* = \boldsymbol{n}, \qquad \boldsymbol{b}^* = \frac{\beta\boldsymbol{b} - \alpha\boldsymbol{t}}{\sqrt{\alpha^2+\beta^2}}. \tag{2.9}$$

The curvature, torsion and arc length of the offset curve are given by

$$\kappa^* = \frac{\beta\kappa - \alpha\tau}{(\alpha^2+\beta^2)\tau}, \qquad \tau^* = \frac{1}{(\alpha^2+\beta^2)\tau}, \qquad \mathrm{d}s^* = \sqrt{\alpha^2+\beta^2}\,\tau\,\mathrm{d}s \tag{2.10}$$

so that the relation

$$\alpha^*\kappa^* + \beta^*\tau^* = 1, \quad \alpha^* = -\alpha, \quad \beta^* = \beta \tag{2.11}$$

shows that the offset curve constitutes another Bertrand curve.

Proof. The orthonormal frame $(2.9)_{2,3,4}$ is obtained from (2.5) and $\boldsymbol{b}^* = \boldsymbol{t}^* \times \boldsymbol{n}^*$. The relations $\boldsymbol{t}^*_{s^*} \cdot \boldsymbol{n}^* = \kappa^*$ and $\boldsymbol{b}^*_{s^*} \cdot \boldsymbol{n}^* = -\tau^*$ provide the expressions $(2.10)_{1,2}$ for the curvature and torsion of the offset curve with $|\mathrm{d}s^*/\mathrm{d}s| = |\boldsymbol{r}^*_s|$ as given by $(2.10)_3$. □

To summarize, a Bertrand curve admits an offset curve at a constant distance α along its principal normal. The offset curve of a Bertrand curve is sometimes called conjugate curve or Bertrand mate. Due to the fact that a Bertrand curve is an offset at a distance $-\alpha$ from its own offset curve, a Bertrand curve and its mate may be regarded as dual to each other. The simplest Bertrand curves and their duals are given by helices.

3. The binormal motion of Bertrand curves. Geodesic Bertrand curves on surfaces

3.1. Razzaboni surfaces

It is well known that a curve Γ constitutes a geodesic on a surface Σ if and only if the principal normal of the curve is (anti-)parallel to the normal $\boldsymbol{N}$ to the surface [26]. This implies that if a surface Σ is spanned by a one-parameter family of geodesic Bertrand curves $\Gamma(b)$ with the same parameters α and β then the Bertrand mates $\Gamma^*(b)$ form a parallel surface Σ^* on which they are likewise geodesics.

Definition 1 (Razzaboni surfaces). A surface Σ is termed a Razzaboni surface if it is spanned by a one-parameter family of geodesic Bertrand curves associated with two constants α and β.

Theorem 2 (Dual Razzaboni surfaces). *Any Razzaboni surface* Σ *with position vector* $\boldsymbol{r}$ *admits a parallel* (*dual*) *Razzaboni surface* Σ^* *with position vector*

$$(\mathcal{R}) \quad \boldsymbol{r}^* = \boldsymbol{r} + \alpha\boldsymbol{n}. \tag{3.1}$$

In the case $\alpha = 0$, *the two Razzaboni surfaces coincide.*

In fact, it is evident that if one demands that a one-parameter family of geodesics on a surface Σ be mapped to geodesics on an offset surface $\Sigma^* : \boldsymbol{r}^* = \boldsymbol{r} + f\boldsymbol{N}$ with $\boldsymbol{N}^* = \boldsymbol{N}$ then, in the generic case, the two surfaces are necessarily parallel and the geodesics constitute Bertrand curves.[1] Accordingly, there exists a complete analogy between classical Bertrand curves and Razzaboni surfaces.

The case $\alpha = 0$ corresponds to surfaces on which there exists a one-parameter family of geodesics of constant torsion. It has been demonstrated in [7] that such surfaces are governed by an integrable extended sine-Gordon system which constitutes a variant of the reduced Maxwell–Bloch equations [8]. Surfaces on which there exists a one-parameter family of geodesics of constant curvature are likewise integrable [7]. The underlying Gauß–Mainardi–Codazzi equations have been shown to reduce to an extended Dym equation which admits a reciprocal invariance. In fact, it emerges that the relation $(2.10)_3$, which links the arc lengths of a Bertrand curve and its conjugate, represents the 'spatial part' of a reciprocal transformation which exists for the Gauß–Mainardi–Codazzi equations of the entire class of Razzaboni surfaces. The reciprocal transformation for the extended Dym equation is retrieved in the particular case $\beta = 0$.

3.2. *The governing equations*

If one chooses a one-parameter family of geodesics and their orthogonal trajectories as the coordinate lines on a surface Σ then, in terms of the associated geodesic coordinates s and b, the first fundamental form of the surface reads [26]

$$\mathrm{d}\boldsymbol{r}^2 = \mathrm{d}s^2 + g^2\,\mathrm{d}b^2. \tag{3.2}$$

Here, the lines $b =$ constant are the arc length parametrized geodesics and the lines $s =$ constant form the orthogonal parallels. Since $\boldsymbol{r}_s \cdot \boldsymbol{r}_b = 0$ and the principal normal $\boldsymbol{n}$ of the geodesics is orthogonal to the surface, the tangent vectors to the coordinate lines are given by

$$\boldsymbol{r}_s = \boldsymbol{t}, \qquad \boldsymbol{r}_b = g\boldsymbol{b}, \tag{3.3}$$

where $\boldsymbol{b}$ denotes the usual binormal of the geodesics. One may therefore think of the surface Σ as being generated by the motion of an inextensible curve which moves in binormal direction at speed g, wherein the coordinate b is identified with 'time'. In particular, a Razzaboni surface is generated by the binormal motion of a Bertrand curve which does not change the constants α and β. Here, it is emphasized that binormal motions are only possible for inextensible curves, i.e. binormal motions automatically preserve arc length.

The variation of the orthonormal triad $(\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b})$ in s-direction is given by the Serret–Frenet equations (2.1). The b-dependence must be of the general form

$$\begin{pmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{pmatrix}_b = \begin{pmatrix} 0 & u & w \\ -u & 0 & v \\ -w & -v & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{pmatrix}. \tag{3.4}$$

[1] Under the assumption that the two surfaces are parallel, this has been observed independently in [27].

The compatibility condition $\boldsymbol{r}_{sb} = \boldsymbol{r}_{bs}$ applied to (3.3) yields

$$u\boldsymbol{n} + w\boldsymbol{b} = -\tau g\boldsymbol{n} + g_s\boldsymbol{b} \tag{3.5}$$

leading to the b-evolution

$$\begin{pmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{pmatrix}_b = \begin{pmatrix} 0 & -\tau g & g_s \\ \tau g & 0 & v \\ -g_s & -v & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{pmatrix}. \tag{3.6}$$

The latter is compatible with the Serret–Frenet equations (2.1) if and only if κ, τ, g and v constitute a solution of the underdetermined system:

$$\kappa_b = -2\tau g_s - \tau_s g, \qquad \tau_b = v_s + \kappa g_s, \qquad g_{ss} = \tau^2 g + \kappa v. \tag{3.7}$$

The above system may be regarded as the Gauß–Mainardi–Codazzi equations for generic surfaces parametrized in terms of geodesic coordinates. For a given solution of this system, the linear system (2.1), (3.3) and (3.6) is compatible and determines a surface Σ up to its position in space. If, in addition, the constraint

$$\alpha\kappa + \beta\tau = 1 \tag{3.8}$$

is imposed then the system is well determined and the surface Σ is guaranteed to be a Razzaboni surface.

In the case $\alpha = 0$, which corresponds to geodesics of constant torsion, we may set $\beta = \tau = 1$ without loss of generality and obtain

$$\kappa_b = -2g_s, \qquad v_s + \kappa g_s = 0, \qquad g_{ss} = g + \kappa v. \tag{3.9}$$

This integrable system may be regarded as an extension of the classical sine-Gordon equation [7]:

$$\omega_{sb} = \sin\omega \tag{3.10}$$

and also constitutes a variant of the reduced Maxwell–Bloch equations [8]. A single equation is obtained by means of the parametrization $\kappa = \theta_s$, $g = -\theta_b/2$, namely

$$\left(\frac{\theta_{bss} - \theta_b}{\theta_s}\right)_s + \theta_s\theta_{bs} = 0. \tag{3.11}$$

If $\beta = 0$, corresponding to geodesics of constant curvature, then $\alpha = \kappa = 1$ without loss of generality and we may set $g = \tau^{-1/2}$. The governing system reduces to the integrable evolution equation [7]:

$$\tau_b = \left[\left(\frac{1}{\tau^{1/2}}\right)_{ss} - \tau^{3/2} + \frac{1}{\tau^{1/2}}\right]_s, \tag{3.12}$$

which represents an extension of the well-known Dym equation

$$\tau_b = \left(\frac{1}{\tau^{1/2}}\right)_{sss}. \tag{3.13}$$

It is noted that the extended Dym equation is generated by the purely binormal motion of an inextensible curve moving at speed $\tau^{-1/2}$.

3.3. Dual Razzaboni surfaces. A reciprocal transformation

It is evident that the transition from Razzaboni surfaces to their duals induces an invariance of the governing equations (3.7) and (3.8). It turns out that the geodesic coordinates (s, b) on Σ are related to the associated geodesic coordinates (s^*, b^*) on Σ^* by a reciprocal transformation. Thus, the following theorem, which constitutes an extension of the second part of the classical Theorem 1, is obtained.

Theorem 3 (A reciprocal transformation). *The nonlinear systems (3.7) and (3.8) are invariant under the reciprocal transformation*

$$\begin{aligned} \mathrm{d}s^* &= \sqrt{\alpha^2+\beta^2}\tau\,\mathrm{d}s + \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}(\alpha v + \beta\tau g + g)\,\mathrm{d}b, \qquad \mathrm{d}b^* = \mathrm{d}b,\\ \kappa^* &= \frac{\beta\kappa - \alpha\tau}{(\alpha^2+\beta^2)\tau}, \qquad \tau^* = \frac{1}{(\alpha^2+\beta^2)\tau}, \qquad \alpha^* = -\alpha,\\ \beta^* &= \beta, \qquad g^* = \frac{\beta(\alpha v + g) - \alpha^2\tau g}{\sqrt{\alpha^2+\beta^2}},\\ v^* &= \frac{1}{\sqrt{\alpha^2+\beta^2}}\left[\beta v - \alpha\tau g - \frac{\alpha}{(\alpha^2+\beta^2)\tau}(\alpha v + \beta\tau g + g)\right]. \end{aligned} \tag{3.14}$$

Proof. It is readily verified that the differentials $\mathrm{d}s^*$ and $\mathrm{d}b^*$ defined by $(3.14)_{1,2}$ are exact modulo (3.7) and (3.8). This guarantees the existence of the coordinates s^* and b^* and hence the corresponding derivatives read

$$\partial_{s^*} = \frac{1}{\sqrt{\alpha^2+\beta^2}\tau}\partial_s, \qquad \partial_{b^*} = \partial_b - \frac{\alpha}{(\alpha^2+\beta^2)\tau}(\alpha v + \beta\tau g + g)\partial_s. \tag{3.15}$$

Differentiation of the dual position vector (3.1) then shows that

$$\boldsymbol{r}^*_{s^*} = \boldsymbol{t}^*, \qquad \boldsymbol{r}^*_{b^*} = g^*\boldsymbol{b}^*, \tag{3.16}$$

where $\boldsymbol{t}^*$ and $\boldsymbol{b}^*$ as given by $(2.9)_{2,4}$ constitute the unit tangent and binormal to the Bertrand curves on Σ^*. Accordingly, s^* represents arc length of the Bertrand curves on Σ^* and b^* parametrizes their orthogonal trajectories. The remaining quantity $v^* = \boldsymbol{n}^*_{b^*}\cdot\boldsymbol{b}^*$ is readily calculated to be $(3.14)_8$. □

In the case $\beta \neq 0$, the reciprocal character of the above invariance encoded in $^{**} = \mathrm{id}$ is illustrated by the compact relations

$$\begin{pmatrix} g^* \\ h^* \end{pmatrix} = S\begin{pmatrix} g \\ h \end{pmatrix}, \qquad S^*S = \mathbb{1}, \tag{3.17}$$

where the constant matrix S is given by

$$S = \frac{\sqrt{\alpha^2+\beta^2}}{\beta}\begin{pmatrix} 1 & \dfrac{\alpha\beta^2}{\alpha^2+\beta^2} \\ \dfrac{\alpha}{\beta^2} & 1 \end{pmatrix} \tag{3.18}$$

and

$$h = v - \frac{\alpha}{\beta}\left(\tau + \frac{1}{\beta}\right) g. \tag{3.19}$$

It is also interesting to note that in the case of surfaces which are spanned by a one-parameter family of geodesic 'generalized helices' ($\alpha, \beta \to \infty, \kappa/\tau = \text{constant}$), the analogue of the above reciprocal transformation linearizes the associated Gauß–Mainardi–Codazzi equations.

4. A linear representation and a Bäcklund transformation

In the derivation of a Bäcklund transformation for Razzaboni surfaces [14], Razzaboni made use of the results of an earlier paper [23] in which he had generalized a Bäcklund transformation for Bertrand curves due to Demartres [24]. Here, we first present the classical theorems set down by Razzaboni and then derive further properties related to the dual Razzaboni surfaces and reciprocal transformation discussed in the preceding. We essentially adopt Razzaboni's notation so that, in particular, the relation between the curvature and torsion of the Bertrand curves is taken to be

$$\kappa \sin\sigma + \tau \cos\sigma = \frac{1}{a}, \quad a > 0, \tag{4.1}$$

i.e.:

$$\alpha = a \sin\sigma, \qquad \beta = a\cos\sigma. \tag{4.2}$$

4.1. Classical results

We begin with Razzaboni's Bäcklund transformation for Bertrand curves.

Theorem 4 (A Bäcklund transformation for Bertrand curves [23]). *Let $\Gamma : \boldsymbol{r} = \boldsymbol{r}(s)$ be a Bertrand curve parametrized in terms of arc length s. Then, the position vector of another one-parameter family*[2] *of Bertrand curves $\Gamma'(k)$ is given by*

$$(\mathcal{B}) \quad \boldsymbol{r}' = \boldsymbol{r} + a\cos k(\cos\sigma \sin\phi\, \boldsymbol{t} + \cos\phi\, \boldsymbol{n} + \sin\sigma\sin\phi\, \boldsymbol{b}) \tag{4.3}$$

with $a' = a, \sigma' = \sigma$, where the function ϕ is a solution of the first-order differential equation

$$\phi_s = \kappa\cos\sigma - \tau\sin\sigma + \frac{\sin\sigma - \cos k\cos\phi}{a(\cos\sigma + \sin k)}. \tag{4.4}$$

The Bäcklund transformation obeys the constant length property, i.e. the distance $|\boldsymbol{r}' - \boldsymbol{r}| = a|\cos k|$ between corresponding points on Γ and $\Gamma'(k)$ only depends on the Bäcklund parameter k.

[2] Strictly speaking, Γ' depends on k and a constant of integration.

It is observed in passing that the above Bäcklund transformation is not defined for $k = -\pi/2 \pm \sigma$. However, in the limit, we formally obtain $|\boldsymbol{r}' - \boldsymbol{r}| = a|\sin\sigma| = |\alpha|$ which is precisely the distance between the original Bertrand curve and its dual. This constitutes a first indication that Bertrand mates may be regarded as particular Bäcklund transforms of their parent Bertrand curves. The surface analogue of the above theorem is the following theorem.

Theorem 5 (A Bäcklund transformation for Razzaboni surfaces [14]). *Let $\Sigma : \boldsymbol{r} = \boldsymbol{r}(s, b)$ be a Razzaboni surface parametrized in terms of geodesic coordinates s, b. Then, the position vector of another one-parameter family of Razzaboni surfaces $\Sigma'(k)$ is given by (4.3), where the function ϕ is a solution of the compatible Frobenius system (4.4) and*

$$\phi_b = -\frac{\sin k \sin\sigma + \cos k \cos\sigma \cos\phi}{\sin k} v - \frac{\sin k \cos\sigma - \cos k \sin\sigma \cos\phi}{\sin k} \tau g - \cot k \sin\phi g_s - \frac{1 + \sin k \cos\sigma - \cos k \sin\sigma \cos\phi}{a \sin k(\cos\sigma + \sin k)} g. \tag{4.5}$$

It is readily verified that the position vector $\boldsymbol{r}'$ and the Frobenius systems (4.4) and (4.5) are invariant under $(k, \phi) \to (\pi - k, \phi + \pi)$. Accordingly, modulo this invariance, the Bäcklund transformation for Razzaboni surfaces exhibits the afore-mentioned 'singularity' at $k = \sigma - \pi/2$ and is likewise undefined for $k = 0$ corresponding to $|\boldsymbol{r}' - \boldsymbol{r}| = a$. In the latter case, it will be shown that consideration of the formal limit $k \to 0$ leads to a Bäcklund transform $\Sigma'(0)$ which is given explicitly in terms of Σ.

The Frobenius systems (4.4) and (4.5) are of the form

$$\phi_s = f_1 \cos\phi + f_2, \qquad \phi_b = g_1 \cos\phi + g_2 + g_3 \sin\phi \tag{4.6}$$

and is therefore equivalent to a pair of compatible Riccati equations. This implies, in turn, that it is linearizable. Indeed, its general solution is given by

$$\phi = 2 \arctan \frac{\phi^1}{\phi^2}, \tag{4.7}$$

where $\Phi = (\phi^1, \phi^2)^{\mathsf{T}}$ obeys the linear system

$$\Phi_s = (f_1 X_1 + f_2 X_2)\Phi, \qquad \Phi_b = (g_1 X_1 + g_2 X_2 + g_3 X_3)\Phi \tag{4.8}$$

with the generators

$$X_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad X_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad X_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.9}$$

of the sl(2) Lie algebra. By construction, the linear system (4.8) is compatible modulo the nonlinear Razzaboni system, i.e. the Gauß–Mainardi–Codazzi equations (3.7) and (3.8). In the terminology of soliton theory, it constitutes a Lax pair for the Razzaboni system with k playing the role of the 'spectral' parameter. In Section 5, it is established that this Lax pair not only encapsulates the Razzaboni system via compatibility but also encodes the Razzaboni surfaces themselves via the Sym–Tafel formula [25].

4.2. A proof of Razzaboni's theorems. Novel results

For the proof of Razzaboni's theorems, it is required to show that the curvature κ' and torsion τ' of the curves Γ' as defined by (4.3) and (4.4) are indeed related by a linear equation of the form (4.1) and that these curves form geodesics on the surfaces Σ'. The latter condition is equivalent to demanding that

$$\boldsymbol{r}'_b \cdot \boldsymbol{n}' = 0, \tag{4.10}$$

which leads to the companion equation (4.5) as shown by Razzaboni. Here, we choose a different route and construct a geodesic coordinate system (s', b') on Σ' which is such that the curves Γ' are given by $b' =$ constant.

4.2.1. Curvature and torsion

Differentiation of the position vector $\boldsymbol{r}'$ as given by (4.3) yields

$$\boldsymbol{r}'_s = f\boldsymbol{t}', \quad f = \frac{1 + \sin k \cos\sigma - \cos k \sin\sigma \cos\phi}{\cos\sigma + \sin k}, \tag{4.11}$$

where the unit tangent $\boldsymbol{t}'$ to Γ' is given by

$$\boldsymbol{t}' = \frac{\cos\sigma + \sin k - \cos k \cos\phi(\sin k \sin\sigma + \cos k \cos\sigma\cos\phi)}{1 + \sin k \cos\sigma - \cos k \sin\sigma\cos\phi}\boldsymbol{t} + \frac{\cos k \sin\phi(\cos k\cos\phi - \sin\sigma)}{1 + \sin k\cos\sigma - \cos k \sin\sigma\cos\phi}\boldsymbol{n} + \cos k\cos\phi\,\boldsymbol{b}. \tag{4.12}$$

Further differentiation yields[3]

$$\boldsymbol{t}'_s = f\kappa'\boldsymbol{n}', \quad \kappa' = \frac{1}{a\sin\sigma} - \cot\sigma\frac{\tau}{f^2}, \tag{4.13}$$

where the principal normal $\boldsymbol{n}'$ reads

$$\boldsymbol{n}' = \frac{\cos k\sin\phi(\sin k\sin\sigma + \cos k\cos\sigma\cos\phi)}{1 + \sin k\cos\sigma - \cos k\sin\sigma\cos\phi}\boldsymbol{t} + \frac{\cos k\cos\phi(\cos k\cos\phi - \sin\sigma) + \sin k(\cos\sigma + \sin k)}{1 + \sin k\cos\sigma - \cos k\sin\sigma\cos\phi}\boldsymbol{n} - \cos k\sin\phi\,\boldsymbol{b}. \tag{4.14}$$

It is noted that for $\sigma = 0$, i.e. $\tau = 1/a$, the above expression for κ' is still valid. In fact, in this case, it simplifies to

$$\kappa' = \kappa + 2\frac{\sin k - 1}{a\cos k}\cos\phi \quad (\sigma = 0). \tag{4.15}$$

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[3] In Section 4, extensive use of the computer algebra program MAPLE has been made. Razzaboni's lengthy calculations which were carried out be hand have thereby been verified.

In order to derive the torsion τ', it is convenient to make use of the binormal $\boldsymbol{b}' = \boldsymbol{t}' \times \boldsymbol{n}'$ given by

$$\boldsymbol{b}' = \frac{\cos k(\cos k \sin\sigma - \sin k \cos\sigma \cos\phi - \cos\phi)}{1 + \sin k \cos\sigma - \cos k \sin\sigma \cos\phi}\boldsymbol{t} + \frac{\cos k \sin\phi(\cos\sigma + \sin k)}{1 + \sin k \cos\sigma - \cos k \sin\sigma \cos\phi}\boldsymbol{n} + \sin k\, \boldsymbol{b}, \tag{4.16}$$

whence

$$\boldsymbol{b}'_s = -f\tau'\boldsymbol{n}', \qquad \tau' = \frac{\tau}{f^2}. \tag{4.17}$$

Combination of $(4.13)_2$ and $(4.17)_2$ yields

$$\kappa' \sin\sigma + \tau' \cos\sigma = \frac{1}{a} \tag{4.18}$$

so that the curves Γ' indeed constitute a family of Bertrand curves with a and σ unchanged.

An important consequence of the preceding which may be regarded as an analogue of a well-known property associated with the classical Bäcklund transformation for pseudospherical surfaces [26,28] is stated below.

Corollary 1. *The angle between the binormals* $\boldsymbol{b}$ *and* $\boldsymbol{b}'$ *of Bertrand curves and their Bäcklund transforms is constant, viz.*:

$$\boldsymbol{b}' \cdot \boldsymbol{b} = \sin k. \tag{4.19}$$

4.2.2. Geodesic coordinates

The relation (4.11) implies that

$$s'_s = f = a\tau + a \sin\sigma \phi_s, \tag{4.20}$$

where s' denotes arc length of the Bertrand curves Γ'. On the other hand, from $(2.10)_3$, we deduce that

$$s^*_s = a\tau = 1 - a \sin\sigma \phi_s|_{k=\pi/2} \tag{4.21}$$

so that the arc lengths of the Bäcklund transform Γ' and the Bertrand mate Γ^* may be expressed in terms of the arc length s of Γ and the function ϕ. Moreover, if $s^{*'}$ denotes arc length of the Bertrand curves on the mate $\Sigma^{*'}$ of the Razzaboni transform Σ' then

$$s'^*_s = s'^*_{s'} s'_s = a\tau' f = \frac{a\tau}{f} = 1 - \frac{a \sin\sigma}{f}\phi_s = 1 - a \sin\sigma \phi^*_s, \tag{4.22}$$

where the function ϕ^* is defined by

$$\phi^* = \int \frac{1}{f} \mathrm{d}\phi = 2 \arctan\left(\frac{1 + \sin(k+\sigma)}{\cos\sigma + \sin k} \tan\frac{\phi}{2}\right). \tag{4.23}$$

Thus, the following theorem is suggested and indeed holds.

Theorem 6 (Geodesic coordinate systems on Razzaboni transforms and Razzaboni mates). *The pairs* (s', b'), (s^*, b^*) *and* (s'^*, b'^*) *defined by*

$$\begin{aligned} &s' = s^* + a\sin\sigma\phi, \quad b' = b,\\ &s^* = s - a\sin\sigma\phi|_{k=\pi/2}, \quad b^* = b,\\ &s^{*'} = s - a\sin\sigma\phi^*, \quad b'^* = b, \end{aligned} \tag{4.24}$$

where ϕ^* *is defined as in* (4.23), *constitute geodesic coordinates on the Razzaboni transform* Σ', *the Razzaboni mate* Σ^* *and the mate* Σ'^* *of the Razzaboni transform* Σ', *respectively*.

Proof. The above theorem which, in conjunction with Theorem 4, implies Theorem 5 is proven by direct verification. In the case of the dual Razzaboni surface, it is readily verified that $\mathrm{d}s^*$ coincides with the expression provided by $(3.14)_1$. Moreover, it has already been demonstrated that $\boldsymbol{r}'_{s'} = \boldsymbol{t}'$. Hence, in connection with the Razzaboni transform Σ', it remains to show that

$$\boldsymbol{r}'_{b'} = g'\boldsymbol{b}' \tag{4.25}$$

for some function g'. It turns out that $\boldsymbol{r}'_{b'}$ is indeed parallel to $\boldsymbol{b}'$ with

$$g' = \frac{g}{\sin k} + a\cot k\cos\sigma(\sin\phi g_s + \cos\sigma\cos\phi h), \tag{4.26}$$

where the function h is defined by (3.19). It is emphasized that even though h is singular at $\sigma = \pi/2$, the quantity g' is well defined for all values of σ. Finally, it is required to verify that $\boldsymbol{r}'^*_{b'^*}$ is parallel to $\boldsymbol{b}'^*$, where the position vector of the surface Σ'^* is given by

$$\boldsymbol{r}'^* = \boldsymbol{r}' + a\sin\sigma\boldsymbol{n}' \tag{4.27}$$

and the associated frame reads (cf. (2.9))

$$\boldsymbol{t}'^* = \cos\sigma\boldsymbol{t}' + \sin\sigma\boldsymbol{b}', \qquad \boldsymbol{n}'^* = \boldsymbol{n}', \qquad \boldsymbol{b}'^* = -\sin\sigma\boldsymbol{t}' + \cos\sigma\boldsymbol{b}'. \tag{4.28}$$

A straightforward computation shows that this is indeed the case. □

4.2.3. A commutativity theorem

It turns out that ϕ^* represents a solution of the Frobenius systems (4.4) and (4.5) associated with the Razzaboni mate Σ^*. Accordingly, application of Theorem 6 to the surface Σ^* produces

$$s^{*'} = s^{**} + a\sin\sigma^*\phi^* = s - a\sin\sigma\phi^* = s'^*. \tag{4.29}$$

In fact, a short calculation reveals that the surfaces Σ'^* and $\Sigma^{*'}$ defined by (4.27) and

$$\boldsymbol{r}^{*'} = \boldsymbol{r}^* + a\cos k(\cos\sigma\sin\phi^*\boldsymbol{t}^* + \cos\phi^*\boldsymbol{n}^* - \sin\sigma\sin\phi^*\boldsymbol{b}^*), \tag{4.30}$$

respectively, coincide. This is summarized in the following statement.

Theorem 7 (A commutation property). *The operations* $\mathcal{B}$ *and* $\mathcal{R}$ *commute, i.e.*:

$$\mathcal{B}\circ\mathcal{R} = \mathcal{R}\circ\mathcal{B} \tag{4.31}$$

provided that the associated functions ϕ *and* ϕ^* *are related by* (4.23).

4.2.4. An invariance of the Razzaboni system

Razzaboni's transformation $\mathcal{B}$ induces an invariance of the Razzaboni system (3.7) and (3.8). We have already established how $\mathcal{B}$ acts on the variables κ, τ, g and s, b. In principal, the remaining quantity v' may be calculated from $v' = \boldsymbol{n}'_{b'} \cdot \boldsymbol{b}'$. However, the expression so obtained appears to be of formidable complexity except when $\sigma = 0$, in which case it reduces to

$$v' = \left(\sin^2\phi + \frac{\cos^2\phi}{\sin k}\right) v + \frac{\cot k}{a}\cos\phi\, g + \left(\frac{1}{\sin k} - 1\right)\sin\phi\cos\phi\, g_s. \quad (4.32)$$

An alternative route to the derivation of v' makes use of the commutativity theorem. Thus, the latter implies that the relation between the variables g', h' and $g^{*'}$, $h^{*'}$ is given by the starred version of (3.17) and (3.18). In particular, the relation

$$g^{*'} = \frac{g'}{\cos\sigma} + a\sin\sigma\cos\sigma\, h' \quad (4.33)$$

obtains. Insertion of h' as given by the primed version of (3.19), i.e.:

$$h' = v' - \tan\sigma\left(\tau' + \frac{1}{a\cos\sigma}\right) g', \quad (4.34)$$

therefore, leads to

$$v' = \frac{g^{*'}}{a\sin\sigma\cos\sigma} + \left(\tan\sigma\,\tau' - \frac{1}{a\sin\sigma}\right) g', \quad (4.35)$$

where g' and $g^{*'}$ are given by (4.26) and its starred analogue, respectively. Once again, it is noted that the above expression for v' is also valid for $\sigma = \pi/2$. Hence, we may conclude this section with the following corollary.

Corollary 2 (An invariance of the Razzaboni system). *The Razzaboni system* (3.7) and (3.8) *is invariant under the transformation* $(\kappa, \tau, g, v, s, b) \to (\kappa', \tau', g', v', s', b')$, *where the primed variables are given by* $(4.13)_2$, $(4.17)_2$, $(4.24)_{1,2}$, (4.26) and (4.35) (*or* (4.32)) *with* $a' = a$, $\sigma' = \sigma$.

4.2.5. A scalar invariant

Razzaboni noticed that for non-vanishing α and β the Gauß–Mainardi–Codazzi equations may be written as a single equation for a potential θ defined by

$$\mathrm{d}\theta = \beta\sqrt{\tau}\,\mathrm{d}s + \alpha\sqrt{\tau}g\,\mathrm{d}b \quad (4.36)$$

and associated with the 'conservation law'

$$(\beta\sqrt{\tau})_b = (\alpha\sqrt{\tau}g)_s. \quad (4.37)$$

Indeed, the conservation law may be used to express τ and g in terms of θ, while $(3.7)_3$ serves as a definition of v. The remaining equation $(3.7)_2$ then constitutes a fourth-order equation for θ. In the case of the extended Dym equation (3.12) corresponding to $\beta = 0$, the analogue of the above exact one-form reads

$$d\theta = \sqrt{\tau}\, ds + \frac{1}{2}\left[\frac{1}{\sqrt{\tau}}\left(\frac{1}{\sqrt{\tau}}\right)_{ss} - \frac{1}{2}\left(\frac{1}{\sqrt{\tau}}\right)_{s}^{2} - \frac{3}{2}\tau + \frac{1}{2\tau}\right] db \tag{4.38}$$

so that the extended Dym equation assumes the form

$$\theta_b = -\frac{\{\theta; s\}}{2\theta_s^2} - \frac{3}{4}\theta_s^2 + \frac{1}{4\theta_s^2} \tag{4.39}$$

with the Schwarzian derivative

$$\{\theta; s\} = \frac{\theta_{sss}}{\theta_s} - \frac{3}{2}\left(\frac{\theta_{ss}}{\theta_s}\right)^2. \tag{4.40}$$

In [7], it has been shown that if θ and $t = -b/2$ are taken as the independent variables then s becomes a potential obeying

$$ds = e^{q}\, d\theta + [e^{q}(q_{\theta\theta} - \tfrac{1}{2}q_\theta^2) - \tfrac{3}{2}\, e^{-q} + \tfrac{1}{2}\, e^{3q}]\, dt, \tag{4.41}$$

where $\tau = e^{-2q}$, and the associated compatibility condition produces the modified modified Korteweg-de Vries (m^2KdV) equation [29,30]

$$q_t = q_{\theta\theta\theta} - \tfrac{1}{2}q_\theta^3 + 3q_\theta \cosh 2q. \tag{4.42}$$

Moreover, it has been demonstrated that the coordinates θ and t are preserved by (what now turns out to be a particular case of) the Razzaboni transformation $\mathcal{B}$ and the reciprocal transformation $\mathcal{R}$. Thus, θ regarded as a function of the geodesic coordinates s and b is an invariant of these transformations. In fact, this property exists in the following general case.

Theorem 8 (A scalar invariant). *The potential θ defined by the exact one-form (4.36) is preserved by the transformations $\mathcal{B}$ and $\mathcal{R}$.*

The above theorem may be verified directly by merely using the expressions for the transformed quantities obtained in the preceding sections. In the case of the duality transformation $\mathcal{R}$, it is required to show that

$$d\theta^* = \beta^*\sqrt{\tau^*}\, ds^* + \alpha^*\sqrt{\tau^*}g^*\, db^* = d\theta. \tag{4.43}$$

In connection with Razzaboni's transformation $\mathcal{B}$, it is convenient to be aware of the relation

$$g' = s'_s g - \cot\sigma\, s'_b, \tag{4.44}$$

which readily delivers the identity

$$d\theta' = \beta\sqrt{\tau'}\, ds' + \alpha\sqrt{\tau'}g'\, db' = d\theta. \tag{4.45}$$

Whether the invariant θ is of any geometric significance is currently under investigation.

4.3. The singular cases $k = \sigma - \pi/2$ and $k = 0$

Razzaboni's Bäcklund transformation $\mathcal{B}$ is not defined for $k = \sigma - \pi/2$ and $k = 0$ since the Frobenius systems (4.4) and (4.5) is singular at these points. However, if formal limits are taken then the duality transformation $\mathcal{R}$ may be recovered from $\mathcal{B}$ in the case $k = \sigma - \pi/2$ and a novel Bäcklund transformation for Razzaboni surfaces may be defined explicitly in the case $k = 0$ corresponding to orthogonal binormals $\boldsymbol{b}$ and $\boldsymbol{b}'$. The validity of the latter transformation may be verified directly without referring to the limit. A rigorous treatment of both limiting procedures will be presented elsewhere.

4.3.1. The case $k = \sigma - \pi/2$

Careful inspection of (4.4) shows that, in the case $k \to \sigma - \pi/2$, it is appropriate to set

$$k = \sigma - \tfrac{1}{2}\pi + \epsilon^2. \tag{4.46}$$

If we indicate the dependence of ϕ on ϵ by $\phi(\epsilon)$ and use the notation

$$\phi_0 = \phi(0), \qquad \phi_1 = \phi_\epsilon(0) \tag{4.47}$$

then, in the formal limit $\epsilon \to 0$, (4.4) reduces to

$$0 = \frac{\phi_1^2}{2a} - \frac{\tau}{\sin\sigma} \tag{4.48}$$

provided that

$$\phi_0 = 0. \tag{4.49}$$

It turns out that these conditions on ϕ_0 and ϕ_1 are consistent with the companion equation (4.5). Moreover, as $\epsilon \to 0$, the transformation laws (4.3), (4.12), (4.14) and (4.16) become

$$\boldsymbol{r}' = \boldsymbol{r}^*, \qquad \boldsymbol{t}' = \boldsymbol{t}^*, \qquad \boldsymbol{n}' = -\boldsymbol{n}^*, \qquad \boldsymbol{b}' = -\boldsymbol{b}^*, \tag{4.50}$$

while, on use of (4.48), the curvature and torsion simplify to

$$\kappa' = -\kappa^*, \qquad \tau' = \tau^*. \tag{4.51}$$

Thus, up to a change of orientation of the principal normal and binormal corresponding to $\alpha' = \alpha = -\alpha^*$, the duality transformation $\mathcal{R}$ is retrieved.

4.3.2. The case $k = 0$

According to Corollary 1, the angle between the binormal $\boldsymbol{b}$ and its Bäcklund transform $\boldsymbol{b}'$ is constant. Since $\boldsymbol{b}' \cdot \boldsymbol{b} = \sin k$, the binormals may even be orthogonal in the case of Bertrand curves. However, the Bäcklund transformation for Razzaboni surfaces is a priori undefined for $k = 0$. If we set

$$k = \epsilon \tag{4.52}$$

and use the same notation as in the preceding then a necessary condition for the existence of a formal limit is that the numerator of the right-hand side of (4.5) vanishes as $\epsilon \to 0$. This implies that

$$a\cos\sigma\, g_s \sin\phi_0 + a\cos^2\sigma\, h\cos\phi_0 + g = 0. \tag{4.53}$$

Here, we exclude the cases $\sigma = \pm\pi/2$ which are covered by the preceding section. The above condition may only be satisfied if

$$a^2\cos^2\sigma g_s^2 + a^2\cos^4\sigma h^2 - g^2 = c^2 \geq 0 \tag{4.54}$$

in which case its solution is given by

$$\sin\phi_0 = -\frac{a\cos\sigma(c\cos\sigma h + gg_s)}{c^2+g^2}, \qquad \cos\phi_0 = \frac{a\cos\sigma(cg_s - \cos\sigma gh)}{c^2+g^2}. \tag{4.55}$$

Differentiation with respect to s shows that (4.54) constitutes a first integral of the Razzaboni system (3.7) and (3.8) with

$$c = c(b) \tag{4.56}$$

and, remarkably, ϕ_0 as defined by (4.55) is indeed a particular solution of the s-evolution (4.4) for $\epsilon = 0$.

In the formal limit $\epsilon \to 0$, the b-evolution (4.5) is readily shown to reduce to

$$\phi_{0b} = -\frac{c\phi_1}{a\cos\sigma} - \frac{\tau g}{\cos\sigma} - \frac{c\tan\sigma gg_s + c^2\sin\sigma h}{c^2+g^2}, \tag{4.57}$$

while the expression (4.26) for g' becomes

$$g' = c\phi_1. \tag{4.58}$$

One may now directly verify that

$$\boldsymbol{r}'_{b'} = g'\boldsymbol{b}', \tag{4.59}$$

where $\boldsymbol{r}', \boldsymbol{b}'$ are evaluated at $\epsilon = 0$ and $\phi = \phi_0$. Thus, the following theorem has been established.

Theorem 9 (A Bäcklund transformation with the property $\boldsymbol{b}'\cdot\boldsymbol{b} = 0$). *If Σ constitutes a Razzaboni surface with associated first integral* $(4.54)_{c^2>0}$ *then the position vector of another Razzaboni surface Σ' is explicitly given by*

$$\boldsymbol{r}' = \boldsymbol{r} + a(\cos\sigma\sin\phi_0\boldsymbol{t} + \cos\phi_0\boldsymbol{n} + \sin\sigma\sin\phi_0\boldsymbol{b}), \tag{4.60}$$

where ϕ_0 is defined by (4.55). *At corresponding points, the binormals $\boldsymbol{b}$ and $\boldsymbol{b}'$ are orthogonal. The metric on Σ' takes the form*

$$\mathrm{d}\boldsymbol{r}'^2 = \mathrm{d}s'^2 + c^2\phi_1^2\,\mathrm{d}b'^2 \tag{4.61}$$

with the quantity ϕ_1 defined by (4.57).

In the particular case $\sigma = 0$ associated with the generalized sine-Gordon system (3.9), the above theorem has been formulated earlier in [7]. We observe in passing that the angle between the normals to a pseudospherical surface and its classical Bäcklund transform [26,28] is likewise constant and may be chosen arbitrarily. In the case of orthogonal normals, Bianchi's classical transformation [31] is retrieved. However, Bianchi's transformation is not given explicitly and depends on a constant of integration.

5. Application of the Sym–Tafel formula

In Section 4.1, it has been pointed out that Razzaboni's Frobenius systems (4.4) and (4.5) are equivalent to the linear system (4.8). We may now set aside the origin of the latter and consider linear systems of the same form but regard Φ as a (complex) matrix-valued 'eigenfunction' satisfying

$$\Phi_s = F(k)\Phi = (f_1X_1 + f_2X_2)\Phi, \qquad \Phi_b = G(k)\Phi = (g_1X_1 + g_2X_2 + g_3X_3)\Phi \tag{5.1}$$

with the coefficients

$$f_1 = -\frac{\cos k}{a(\cos\sigma + \sin k)}, \qquad f_2 = \kappa\cos\sigma - \tau\sin\sigma + \frac{\sin\sigma}{a(\cos\sigma + \sin k)},$$
$$g_1 = \cot k\left(-\cos\sigma\, v + \sin\sigma\,\tau g + \frac{\sin\sigma}{a(\cos\sigma + \sin k)}g\right),$$
$$g_2 = -\sin\sigma\, v - \cos\sigma\,\tau g - \frac{1 + \sin k\cos\sigma}{a\sin k(\cos\sigma + \sin k)}g, \qquad g_3 = -\cot k g_s \tag{5.2}$$

and the sl(2) matrices

$$X_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad X_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad X_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.3}$$

Here, we have indicated the dependence on the (complex) parameter k by $F = F(k)$ and $G = G(k)$. By construction, this linear system encapsulates via compatibility the nonlinear Razzaboni system (3.7) and (3.8). Since the latter constitutes the Gauß–Mainardi–Codazzi equations for Razzaboni surfaces which, in turn, guarantee the compatibility of the linear Gauß–Weingarten equations for the position vector $\boldsymbol{r}$, it is natural to enquire as to whether the Razzaboni surfaces themselves are encoded in the Lax pair (5.1).

We begin with the observation that the specification

$$k = 2\arctan(\mathrm{i}a\lambda) - \tfrac{1}{2}\pi, \tag{5.4}$$

where λ constitutes a real parameter, results in real coefficients f_2, g_2 and purely imaginary coefficients f_1, g_1, g_3. Thus, the matrices F and G are now elements of the Lie algebra su(2) since

$$F^\dagger = -F, \qquad G^\dagger = -G \tag{5.5}$$

and Φ may be taken to be in the associated Lie group SU(2) obeying

$$\Phi^\dagger\Phi = \mathbb{1}, \qquad \det\Phi = 1. \tag{5.6}$$

At $\lambda = 1/a$, i.e. $k \to \mathrm{i}\infty$, the Lax pair reduces to

$$\Phi_s = (-\tau\hat{e}_1 - \kappa\hat{e}_3)\Phi, \qquad \Phi_b = (-v\hat{e}_1 + g_s\hat{e}_2 + \tau g\hat{e}_3)\Phi, \tag{5.7}$$

where the matrices $\hat{e}_i$ are given by

$$\hat{e}_1 = \cos\sigma\, e_1 - \sin\sigma\, e_2, \qquad \hat{e}_2 = -e_3, \qquad \hat{e}_3 = \sin\sigma\, e_1 + \cos\sigma\, e_2 \tag{5.8}$$

with the standard generators

$$e_1 = \frac{1}{2\mathrm{i}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad e_2 = \frac{1}{2\mathrm{i}}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad e_3 = \frac{1}{2\mathrm{i}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5.9}$$

of su(2). Both sets of matrices e_i and $\hat{e}_i$ satisfy the so(3) commutator relations

$$[e_i, e_k] = \epsilon^l_{ik} e_l. \tag{5.10}$$

Moreover, the Gauß–Weingarten equations (2.1) and (3.6) may be cast precisely into the form (5.7) with the substitutions $\Phi \to (\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b})^\top$ and $\hat{e}_i \to l_i$, where the generators l_i of the Lie algebra so(3) are defined by

$$l_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad l_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad l_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.11}$$

and satisfy the commutator relation (5.10). Thus, the Lax pair (5.1) evaluated at $\lambda = 1/a$ is but an su(2) version of the Gauß–Weingarten equations for Razzaboni surfaces by virtue of the su(2)–so(3) isomorphism $\hat{e}_i \leftrightarrow l_i$.

Since $\Phi \in \mathrm{SU}(2)$, the quantity $\Phi^{-1}\Phi_\lambda$ represents an element of su(2) and hence may be decomposed according to

$$R = \Phi^{-1}\Phi_\lambda = \boldsymbol{r} \cdot \boldsymbol{e}, \qquad \boldsymbol{e} = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{5.12}$$

Thus, the matrix-valued function R is naturally associated with a vector-valued function $\boldsymbol{r} \in \mathbb{R}^3$ defined by (5.12) or equivalently

$$\boldsymbol{r} = m(R, \boldsymbol{e}), \tag{5.13}$$

where the Killing–Cartan metric of su(2) is given by

$$m(p, q) = -2\,\mathrm{Tr}(pq), \qquad p, q \in \mathrm{su}(2). \tag{5.14}$$

It is noted that $\{e_1, e_2, e_3\}$ and $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ constitute orthonormal bases of su(2) with respect to m, i.e.:

$$m(e_i, e_k) = \delta_{ik}. \tag{5.15}$$

The relation $R = \Phi^{-1}\Phi_\lambda$ is commonly referred to as the ‘Sym–Tafel formula’ and has been widely employed in connection with the geometric study of both continuous and discrete integrable systems [1,25,32]. The key idea is to identify the Lie algebra su(2) with $\mathbb{R}^3$ and regard $\boldsymbol{r}$ as the position vector of a surface $\Sigma \subset \mathbb{R}^3$ for any fixed λ.

In the current situation, it is by no means evident that the surfaces Σ defined by the Sym–Tafel formula constitute Razzaboni surfaces for any choice of λ. However, it turns out that at $\lambda = 1/a$, Razzaboni surfaces are indeed retrieved. Thus, in the following, it is understood that all relevant quantities are evaluated at $\lambda = 1/a$. The tangent vectors to the coordinate lines on Σ are readily obtained from the general relations

$$R_s = \Phi^{-1} F_\lambda \Phi, \qquad R_b = \Phi^{-1} G_\lambda \Phi, \tag{5.16}$$

which imply that the associated fundamental forms are independent of the eigenfunction Φ and are entirely parametrized in terms of the matrices F and G. In the present context, the su(2) analogues of the tangent vectors are given by

$$R_s = \Phi^{-1} \hat{e}_1 \Phi, \qquad R_b = g \Phi^{-1} \hat{e}_3 \Phi. \tag{5.17}$$

Since the quantities

$$T = \Phi^{-1} \hat{e}_1 \Phi, \qquad N = \Phi^{-1} \hat{e}_2 \Phi, \qquad B = \Phi^{-1} \hat{e}_3 \Phi \tag{5.18}$$

form a right-handed orthonormal triad with respect to the metric m, it emerges that this triad is nothing but an su(2) analogue of the $(\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b})$-frame associated with the curves $b =$ constant and the induced metric assumes the 'geodesic' form

$$\mathrm{d}\boldsymbol{r}^2 = \mathrm{d}s^2 + g^2\, \mathrm{d}b^2. \tag{5.19}$$

Moreover, differentiation of T and B produces

$$T_s = \kappa N, \qquad B_s = -\tau N. \tag{5.20}$$

Thus, we have established the important result that the position vector of any Razzaboni surface Σ may be recovered and, in fact, constructed from the eigenfunction Φ by means of the Sym–Tafel formula (5.12) evaluated at $\lambda = 1/a$.

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